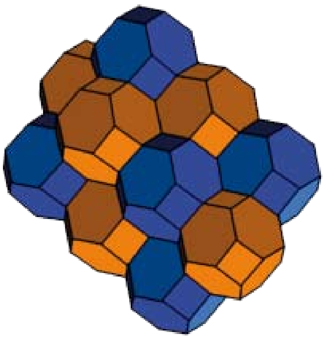
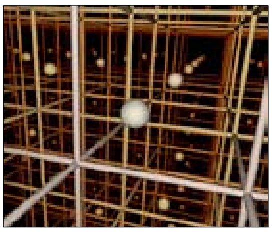
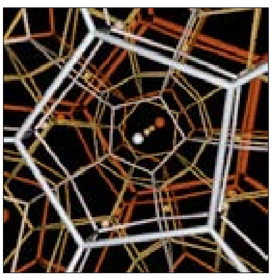
Tiling in curved space

By Jenny Rankine

Above: Gaven Martin; photo courtesy of Massey University. **Immediately below: A bi-truncated cubic honeycomb**







Professor Gaven Martin is in full flight, describing how non-linear partial differential equations explain how the microstructure in a steel bar affects the macrosopic distribution of force as it is bent. If his listener doesn't understand, he takes a step back and sketches the graph of a simple function on his office whiteboard at Massey University in Albany.

He shows how finding its minima leads to a differential equation. The *x* and *y* axes no longer represent single variables, but states – which are functions in themselves. So the energy of the bar is now a function of functions. The solution of the partial differential equation (PDE) will be the particular bend that, out of all possible twists, uses the least energy.

This PDE might be solvable for a simple steel bar, but for most modern materials and structures, such as cars, bridges, tower blocks and even bars made from composite materials, other PDEs are not. "So we build models of the key energy functionals of those structures - their crystalline alignment or the interface between two different components - and then minimise them, leading often to quite complicated non-linear PDEs."

"Pure mathematicians study classes of differential equations with generic features to validate these models. Generic features include whether the equation has a solution or not," he says. "This sounds almost useless, but it's not. It's a damn sight better searching for something if you know it's there. Other important features include whether there are multiple solutions or only one and whether the solution is regular or not, because otherwise a computer has no chance of finding it. The solutions to differential equations can be very irregular."

In 1850 Joseph Liouville hypothesised that the only conformal transformations of space (the symmetries of physical theories such as relativity) are the usual Lorentz or Mōbius transformations. Liouville proved this if the symmetry was sufficiently smooth. "But this begs the question," says Martin. "Physical systems exist where solutions are highly irregular. Liouville was effectively asking if there were any physical symmetries other than the obvious ones." In 1990, with Polish mathematician Tadeusz Iwaniec, Martin completely solved this problem.

"We showed *exactly* how regular a solution would have to be for it to be a Mōbius transformation; we also constructed solutions, functions, that weren't. But they were so irregular that they couldn't represent a physical symmetry – they blew up all over the place." This result remains the only fully non-linear system of PDEs in more than two dimensions with an exact regularity theory. The tools and ideas created in the proof (non-linear Hodge theory) opened a theoretical bottleneck and led to hundreds of mathematical papers being published.

Martin's description of a new result in hyperbolic (curved space) geometry leads to another flurry of whiteboard drawings. This time they are spheres, tori (donut shapes) and other surfaces with more holes and the two-dimensional polygons that tessellate their surfaces.

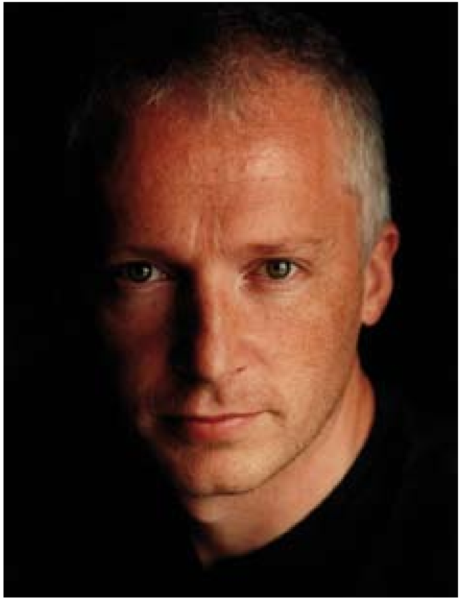
Tessellations are periodic tilings of two or more dimensions that cover a plane, like graph paper, or fill a multi-dimensional shape, like honeycomb, with no overlaps and no gaps. The result Martin is describing is the formula for the smallest possible three-dimensional tessellation of hyperbolic space – solving a problem first posed by Carl Siegel in 1945 after he had solved the 2D problem.

Every surface can be identified with a hyperbolic tessellation and recently results of Grisha Perelman (of Poincaré conjecture fame) show the same is more or less true in three dimensions. Martin, with Fred Gehring and former PhD student Tim Marshall, showed that tetrahedrons with internal angles $\pi/2$, $\pi/3$ and $\pi/5$ gave the smallest possible 3-D hyperbolic tessellation, and found deep connections with number theory.

"Sometimes these simple problems take decades." Martin worked on the 3-D tessellation problem for nearly 20 years, getting partial results in small steps. "You get the occasional blinding flashes of insight, but they often come when you're younger."

Martin met his future wife Dianne, a biology professor at Massey University, when they attended Auckland's Henderson Intermediate. He did his PhD at the University of Michigan and has worked in the USA, Finland, Sweden, France and Australia. He has recently become director of the new NZ Institute for Advanced Study. Martin was one of the NZIMA's original principal investigators, and represents the New Zealand Mathematics Research Institute (Inc.) on the NZIMA board.

Prime promoter



Soccer player, musician and mathematician Marcus du Sautoy, a 2007 NZIMA Maclaurin Fellow, spoke with Jenny Rankine.

Marcus du Sautoy is 41 and bought a house at number 53 in a North London street. He can take the 19 or 73 bus to get there or drive his Primera 47. In 2003 when he was 37 he wrote a bestselling book which has been translated into 11 languages. All these numbers might tell you the subject - prime numbers - which he calls "the atoms of mathematics".

Despite buying his Sunday soccer team prime number t-shirts, he found it "freaky" that his subconscious had filled his life with so many primes.

Du Sautoy is that walking contradiction, a popular mathematician. He is a professor of mathematics at Oxford, and enjoys a media fellowship which enables him to spend half his time in the abstruse world of theoretical symmetry in infinite dimensions, and the other half popularising mathematics. This involves him in a bewildering array of media and collaborations.

He presents the BBC television program, Mind Games, where contestants solve mathematical and language problems. He has made short videos for school students; his four one-hour BBC maths history documentaries will go to air next year; he has a radio series and newspaper columns.

Num8er My5teries

In 2006 he scored the ultimate TV scientist slot - the Royal Institution Christmas Lectures - which drew over a million viewers. He called them The Num8er My5teries, about five of the Clay Mathematics Institute's Millenium Problems, and will repeat them in Australia in the near future. He collaborated on the play A Disappearing Number, about mathematicians Srinivasa Ramanujan and GH Hardy, which is touring the UK.

Du Sautoy has also worked with New Zealand dance choreographer Carol Brown and composer Dorothy Ker as part of the UK's burgeoning sci-art movement. "We combined maths, music, sculpture and choreography to explore ideas of infinity and space." He finds these collaborations very stimulating: "they help me look at things in new ways."

He's enthusiastic about the drama behind his supposedly dry subject - his book The Music of the Primes is full of near escapes from death, love, betrayal and mad genius, proofs scribbled in margins and published from prison. When I meet him, he's wearing a purple soccer t-shirt with a prime number on the back and the equation describing the trajectory of a ball on the front. "Every time you catch or kick a ball, you're solving this quadratic equation," he says.

Du Sautoy describes the elation of discovering something first, which mathematicians crave for the rest of their careers. His discovery was a symmetry group. "Before I found it, it didn't seem to exist at all. Like an alchemist, I put things together in unexpected ways and pushed the subject in a new direction. Afterwards it felt like it had been lying there like a piece of gold waiting to be discovered."

His talent with metaphor is used to the full in his book The Music of the Primes, which describes the fiendishly complicated Riemann hypothesis in terms of harmony and landscape. Both the book and du Sautoy have fun websites. He has a new book coming out in early 2008 called Finding Moonshine, which tells the story of symmetry.

During his visit earlier this year, he spent a week with Ben Martin, a group theorist at the University of Canterbury. "A week's a short time, but we were able to connect very quickly." In the competitive world of elite mathematics, good collaborations are precious.

Du Sautoy works regularly with a German mathematician, Fritz Grunewald. "It feels like having two ladders, each with rungs missing so we can't climb further alone. As soon as we meet, everything fizzes. Often I'll be standing at a white board, grumbling, and he runs with it. Teasing out what's going on in non-verbal moments is as important as talking."

In Auckland du Sautoy lectured on the primes and made use of the computer skill of Auckland mathematicians on infinite groups. "It was fun doing computer experiments, it felt like being a scientist - I understood the groups a lot better."

He loves to overturn the popular misconception that maths research is long division to a lot of decimal places and that computers must have worked it all out by now. "Computers can help mathematicians generate data in their search for patterns; mathematicians have to try to show why the patterns will always work," he says.

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See also

www.musicoftheprimes.com/ and www.maths.ox.ac.uk/~dusautoy/

For a free DVD of the Christmas Lectures: www.rigb.org/rimain/news/newsdetail.jsp?&comp=1&id=122&lang=EN

Videos on Teachers TV: www.teachers.tv/search/node/du+Sautoy

www.lms.ac.uk/activities/education_com/videos_purchase.html

A DVD of a BBC documentary presented by du Sautoy based on The Music of the Primes is available at www.ouw.co.uk/products/XM002_DVDSERIES.shtm

NOTABLE MATHS PROBLEMS

IS P = NP?

Are complexity classes P and NP equivalent?

Simply: If positive solutions to a yes/no problem can be verified quickly, can they also be computed quickly?

Example: Does some subset of the set {−2, −3, 15, 14, 7, −10} add up to zero? Yes, and it's easy to check that {−2, −3, −10, 15} does, with simple addition. Verifying that a subset adds up to zero can be much faster than finding the subset in the first place. How to find such a subset when the set itself is large is called the subset-sum problem. This is an example of a problem in the class NP (non-deterministic polynomial time). Verifying the answer is an example of a problem in the class P (polynomial time).

Discipline: Computational complexity theory, which deals with the resources, such as time (expressed in the number of steps) and space (expressed as memory), needed to solve a given problem.

Incentive: $US1million, one of the seven Millennium Prize Problems of the USA-based Clay Mathematics Institute.

Unusual aspect: Involves a computer that has never been built. Turing machines, first described by English mathematician and cryptographer Alan Turing in 1936, were a thought experiment to simulate the logic of any computer that could possibly be constructed.

Consequences: Remarkably, a polynomial-time solution to a problem in NP would provide polynomial-time solutions to *all* problems in NP. If P=NP, then this would simplify many currently intractable mathematical problems in fields as varied as operations research, logistics and biology. On the other hand, a proof that $P \neq NP$ would show that many common problems cannot be solved efficiently, moving the attention of researchers to partial solutions or other problems. As experts believe that $P \neq NP$, this is happening already.

NZIMA link: 2004 Logic and Computation programme.

Awards and honours

HYMAN BASS, a visiting Maclaurin Fellow last year, was presented with a National Medal of Science by the President of the USA in July. He was one of eight winners of this award for 2006.

In June, NZIMA Co-Director **VAUGHAN JONES** was awarded the 2007 Prix Mondial Nessim Habif by the University of Geneva, for his achievements in mathematics. This an annual prize awarded to prominent researchers from all disciplines.

ROBERT MCLACHLAN (one of the NZIMA's principal researchers) was awarded the Dahlquist Prize at the SciCADE meeting in St Malo in July 2007, for his outstanding contributions to geometric integration and composition methods for solving differential equations.

CHERYL PRAEGER, a member of the NZIMA's International Scientific Advisory Board, has been awarded a prestigious Federation Fellowship by the Australian Research Council.

JAMES SNEYD, one of the NZIMA's principal researchers, has been awarded a James Cook Fellowship for 2007-2009.

> **Every mathematician worthy of the name has experienced … the state of lucid exaltation in which one thought succeeds another as if miraculously … Once you have experienced it, you are eager to repeat it but unable to do it at will, unless perhaps by dogged work …**
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> *André Weil, 1991*

MATHEMATICAL EVENTS

5-7 November 2007, Long Bay, Auckland
Workshop for Women in the Mathematical Sciences in NZ Contact Dr Vivien Kirk, v.kirk@auckland.ac.nz

22-23 November, Queenstown
NZ Mathematics and Statistics Postgraduate (NZMASP) Conference
www.math.canterbury.ac.nz/bio/NZMASP/ or contact Scott Graybill, sjg146@student.canterbury.ac.nz

30 November - 1 December, Christchurch
41st Annual ORSNZ Conference
https://secure.orsnz.org.nz/conf/

3-7 December, Dunedin
32nd ACCMCC (Australasian Conference on Combinatorial Mathematics and Combinatorial Computing) www.cs.otago.ac.nz/staffpriv/mike/ACCMCC32/32ACCMCC.html

9-11 December, Victoria University of Wellington
Mathematical & Computational Nanoscience 2007 (sponsored jointly by the NZIMA and the MacDiarmid Institute)
www.macdiarmid.ac.nz/mcn/

12-15 December, Victoria University of Wellington
1st Joint Meeting of the American and New Zealand Mathematical Societies www.mcs.vuw.ac.nz/~mathmeet/amsnzms2007/

6-12 January 2008, Nelson
2008 NZMRI Conference on Conformal Geometry
www.math.auckland.ac.nz/wiki/2008_NZMRI_Conference_on_Conformal_Geometry

14-18 January, Kaikoura
Conference on Finite Groups and Representations
www.math.canterbury.ac.nz/bio/Finite_Groups/

28 January - 1 February, Woollongong, Australia
Mathematics in Industry Study Group 2008
www.uow.edu.au/informatics/maths/research/misg/index.html

3-7 February, Katoomba, Australia
ANZIAM 2008
www.maths.usyd.edu.au/ANZIAM2008/

18-22 February, Napier
Conference on Algorithms
www.cs.otago.ac.nz/algorithmics/activities/febmeeting.html

19-22 February, Auckland
Workshop on Multi-scale Modelling of the Respiratory System
www.bioeng.auckland.ac.nz/events/msmrs/index.php